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Unsteady laminar convection in uniformly heated vertical pipes

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In this paper an exact solution is presented for the problem of unsteady laminar convective flow under a pressure gradient along a vertical pipe. We have obtained the solution of the problem on the basis of the assumption that the velocity and buoyancy profiles far from the pipe entrance do not change with the height, and the entry lengths have been ignored. The wall of the pipe is heated or cooled uniformly. We have discussed both the cases, when buoyancy forces act together with the pressure gradient or in opposite direction.

In the case when the upflow is heated (or a downflow is cooled) the velocity and thermal boundary layers are formed for sufficiently large Rayleigh numbers. In the second case which has been discussed in detail (when the upflow is cooled or the downflow is heated) we have found the critical value of the Rayleigh number $R = R_c$ beyond which the velocity profile and the temperature profile become unsteady and turbulent in all the cases. In the case of the elliptical cylinder R_c increases up to 1730 as the ellipticity is increased while in the case of the co-axial pipes this Rayleigh number increases as the gap c between the cylinders is decreased (if $c = a/b = 1{\cdot}2$ then $R_c = 60762$, but decreases to 1 when $c = 4$). Besides this, the time required to reach steady state increases as the Rayleigh number increases in both circular and elliptical pipes; it also increases when the eccentricity is decreased. The cases discussed by Morton (1960) and Dalip Singh (1965) are particular cases of the results derived below.

In this investigation we have dealt with the following ducts: (i) circular tubes, (ii) elliptical tubes and (iii) co-axial tubes. The general solutions for both velocity and temperature fields have been found for the case when the pressure gradient is an arbitrary function of time, with an arbitrary heat source also present. Particular cases when both the parameters are absolute constants have been discussed in detail.

We have made use of finite transforms very frequently; especially for the case of an elliptical tube, a new transform involving Mathieu functions developed by Gupta (1964) has been used. A few new infinite series have been summed with the help of this transform.

Various non-dimensional quantities (for both the cylinders) such as the Nusselt number, volume flux and rate of heat transfer have been found when the pressure gradient and source of heat generation are absolute constants.

1. Introduction

Heat-transfer problems of forced convection in channels have constituted an attractive, important and useful subject of investigation for several years. The free and forced convection problems for channels under fully developed conditions with constant wall temperature have been investigated for many years.

In these problems thermal convection consists of the transport of heat by a moving fluid in which the vertical variations of the temperature (and hence density) produce a distributed buoyancy force that itself modifies the flow. This interaction of velocity and temperature is an essential feature of convection; hence, to find the convection flow in a heated flow, both the temperature and velocity fields must be determined throughout the whole region of flow.

At a first glance it may seem that the problem is simply a change of boundary conditions but solutions of this kind are usually difficult to obtain even in the steady case, and so under the condition only simple categories of forced and natural convection have been studied theoretically in detail.

The energy equation and momentum equation are not uncoupled as in the case of constant wall temperature, hence mathematically this problem is not so simple. This implies that in order to study the velocity and temperature fields in this mixed boundary-value problem it is necessary to seek solutions of an inhomogenous biharmonic equation obtained after eliminating either the velocity or temperature field in the steady case. This approach has been used by many research workers. Tao (1961) has suggested introducing a complex analysis, but this method gives an equation which is very involved in the unsteady case.

The main aim of this investigation is to solve the unsteady case for tubes of various cross-sections by the method of transform calculus. So far as is known to the author the solution of the unsteady case for a circular cylinder has not been treated in such a general manner before, hence it has been investigated in great detail here. Another object of this study is to discuss the problem for a hollow elliptical cylinder with the help of the finite transform (discussed in an earlier paper by the author (1964)). The results obtained by various authors (e.g. Morton 1960; Dalip Singh 1965) are particular cases of the results obtained here.

The striking feature of these investigations is that the Rayleigh number increases as the eccentricity of the hollow elliptical cylinder is increased. Following King & Wiltse (1958) it has been found that the value of the critical Rayleigh number increases up to 1730 when ellipticity is increased while in the case of a circular cylinder it is 33. It has been found further that in the case of co-axial circular pipes R_c increases to 60762 when $c = a/b = 1{\cdot}2$ and decreases to unity when $c = a/b = 4$.

2. Fundamental equations

Consider the unsteady flow in the direction of the axis of a pipe of a fluid with density ρ under a pressure gradient. Since the pressure gradient is not necessarily a constant in unsteady flow, we have assumed it to be an arbitrary function of time along the vertical pipe, the walls of which are maintained at a uniform temperature gradient in the direction of the axis. The system will be referred to in a Cartesian system of co-ordinates, and the axis of the pipe is directed vertically upwards.

The equations of continuity, momentum and energy are written using the Boussinesq approximation, i.e.

$$\partial_j u_j = 0,$$

$$\partial_t u_i + u_j \partial_j u_i = -\frac{1}{\rho_0}\partial_i p - \frac{p}{\rho_0} g\lambda_i + \nu\nabla^2 u_i,$$

$$\partial_t T + u_j \partial_j T = k\nabla^2 T + Q,$$

where we have used the summation convention and the following notation:

$$\partial_j = \partial/\partial x_j, \quad \partial_t = \partial/\partial t \quad (j = 1, 2, 3).$$

$\boldsymbol{\lambda}$ is the unit vector in the direction opposite to that of gravity; Q is the heat source term and in general is a function of the spatial co-ordinates and time. All other symbols have their usual meaning. Suppose that the inner wall of the cylinder is held at temperature T_1, and T_0 is the temperature of the pipe at the level of the origin. Let a be the characteristic length, and let us suppose the temperature to be of the form

$$T - T_0 = -\beta' x_j \lambda_j + \theta,$$

where the first term, in which $\beta' = (T_0 - T_1)/a$, describes the temperature in the static state and θ is the deviation from the linear distribution. The fundamental equations must be supplemented by an equation of state, which we approximate as

$$\rho = \rho_0[1 - \alpha(T - T_0)].$$

Hence we arrive at the well-known system of equations

$$\partial_i u_i = 0, \tag{2.1}$$

$$\partial_t u_i + u_j \partial_j u_i = -\partial_i \tilde{\omega} + \alpha g \theta \lambda_i + \nu \nabla^2 u_i, \tag{2.2}$$

$$\partial_t \theta + u_j \partial_j \theta = \beta \lambda_j u_i + k\nabla^2 \theta + Q_i, \tag{2.3}$$

where

$$\tilde{\omega} = p/\rho_0 + g x_j \lambda_j - \tfrac{1}{2}\beta' \alpha g x'_k \lambda'_k x_j \lambda_j$$

and α, g, ν and k are assumed to be constants.

To obtain the non-dimensional form of the equations we set

$$u_i = \frac{u'_i k}{a}, \quad \theta = \frac{\nu k \theta'}{\alpha g a^3}, \quad t' = \frac{a^2 t}{k}, \quad x_i = a x'_i, \quad \tilde{\omega} = \frac{k^2 \tilde{\omega}'}{a^2}.$$

This yields, after dropping the primes,

$$\partial_j u_j = 0,$$

$$\partial_t u_i + u_j \partial_j u_i = -\partial_i \tilde{\omega} + P\lambda_i \theta + P\nabla^2 u_i,$$

$$\partial_t \theta + u_j \partial_j \theta = R u_i + \nabla^2 \theta + Q_i \alpha_i,$$

where $P = \nu/k$ is the Prandtl number, $R = \alpha g \beta' a^4/k\nu$ is the Rayleigh number and $\alpha_i = \alpha g a^5/k^2 \nu$.

Let the fluid flow be in the direction of the axis of the cylinder and u_3, say, in the direction of the axis of the cylinder, then $u_1 = 0 = u_2$ and the equations are transformed into

$$\partial u_3/\partial x_3 = 0,$$

$$\left.\begin{aligned} \partial u_3/\partial t &= -\partial\tilde{\omega}/\partial x_3 + P\nabla_1^2 u_3 + P\theta, \\ \partial\theta/\partial t &= Ru_3 + \nabla_1^2\theta + Q, \end{aligned}\right\} \tag{2.4}$$

where $\nabla_1^2 = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2.$

In this one-dimensional flow problem, where conditions vary in time but not in the x_3 direction (apart from the pressure gradient), the equations become linear.

Differentiating the first equation of (2.4) with respect to x_3, we find that grad $(\partial\tilde{\omega}/\partial x_3)$ vanishes, hence $\partial\tilde{\omega}/\partial x_3$ is a function of time only. Thus we can write (2.4) in the following form:

$$\partial u_3/\partial t = P\nabla_1^2 u_3 + P\theta - f(t), \tag{2.5}$$

$$\partial\theta/\partial t = Ru_3 + \nabla_1^2\theta + \psi(t)\,g(x_1, x_2), \tag{2.6}$$

where $Q = \psi(t)\,g(x_1, x_2)$.

Equations (2.5) and (2.6) are to be solved under the following boundary conditions:

(i) $u_3 = 0 = \theta$ on rigid boundaries,

(ii) $u_3 = 0 = \theta$ initially.

3. Circular tubes

In this section the case of a circular tube is investigated. Since all the quantities are independent of ϕ and depend only on r owing to axial symmetry, (2.5) and (2.6) in cylindrical polar co-ordinates take the form

$$\left.\begin{aligned} \frac{\partial u_3}{\partial t} &= P\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)u_3 + P\theta - f(t), \\ \frac{\partial\theta}{\partial t} &= Ru_3 + \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\theta + \psi(t)\,g(r), \end{aligned}\right\} \tag{3.1}$$

with the boundary conditions

$$\left.\begin{aligned} u_3(r,0) &= 0 = \theta(r,0) \quad \text{when} \quad t = 0, \\ u_3(1,t) &= 0 = \theta(1,t) \quad \text{for} \quad t > 0. \end{aligned}\right\} \tag{3.2}$$

By multiplying (3.1) by $rJ_0(rq_i)$ and integrating with respect to r between 0 and 1, we get

$$d\bar{u}_3/dt = -Pq_i^2\bar{u}_3 + P\bar{\theta} - f(t)\,J_1(q_i)/q_i, \tag{3.3}$$

$$d\bar{\theta}/dt = R\bar{u}_3 - q_i^2\bar{\theta} + \psi(t)\,\bar{g}(q_i), \tag{3.4}$$

where $\bar{\theta} = \int_0^1 r\theta J_0(q_i r)\,dr; \quad \bar{u}_3 = \int_0^1 ru_3 J_0(rq_i)\,dr$

and q_i is the ith positive root of $\quad J_0(q) = 0, \quad$ (3.5)

with the boundary conditions

$$\bar{u}_3 = 0 = \bar{\theta}, \quad d\bar{u}_3/dt = 0 = d\bar{\theta}/dt \text{ initially.} \tag{3.6}$$

Eliminating $\bar{\theta}$ from the (3.3) and (3.4), we have

$$\frac{d^2\bar{u}_3}{dt^2}+(P+1)q_i^2\frac{d\bar{u}_3}{dt}+(Pq_i^4-RP)\bar{u}_3 = P\bar{g}(q_i)\psi(t)-q_iJ_1(q_i)f(t)-f'(t)\frac{J_1(q_i)}{q_i}$$
$$= \chi(t), \quad \text{say.} \tag{3.7}$$

The appropriate solution of (3.7) with the boundary conditions (3.6) is given by

$$\bar{u}_3 = \frac{F'(0)-\beta F(0)}{\beta-\alpha}e^{\alpha t}+\frac{\alpha F(0)-F'(0)}{\beta-\alpha}e^{\beta t}+F(t), \tag{3.8}$$

where
$$F(t) = e^{\alpha t}\int^t e^{(\beta-\alpha)t}\int^t \chi(t)\,e^{-\beta t}\,(dt)^2$$

and
$$\alpha = \tfrac{1}{2}\{-(P+1)q_i^2+[(P-1)^2q_i^4+4RP]^{\frac{1}{2}}\},$$
$$\beta = \tfrac{1}{2}\{-(P+1)q_i^2-[(P-1)^2q_i^4+4RP]^{\frac{1}{2}}\}.$$

Using the well-known inversion formulae, (3.8) is inverted to give

$$u_3 = 2\sum_i\left[\frac{F'(0)-\beta F(0)}{\beta-\alpha}e^{\alpha t}+F(t)+\frac{-F'(0)+\alpha F(0)}{\beta-\alpha}e^{\beta t}\right]\frac{J_0(rq_i)}{\{J_1(q_i)\}^2}, \tag{3.9}$$

where summation is taken over all the positive roots of (3.5). It is quite obvious that as R (the Rayleigh number) increases the velocity stabilizes, but as soon as R reaches the critical value, i.e. $R = q_1^4$ (q_1 being the first zero of (3.5)) the steady state is disturbed.

Similarly,

$$\bar{\theta} = G(t)+\frac{G'(0)-\beta G(0)}{\beta-\alpha}e^{\alpha t}+\frac{\alpha G(0)-G'(0)}{\beta-\alpha}e^{\beta t}, \tag{3.10}$$

where
$$G(t) = e^{\alpha t}\int e^{(\beta-\alpha)t}\int e^{-\beta t}\,Y(t)\,(dt)^2 \tag{3.11}$$

and
$$Y(t) = Pq_i^2\bar{g}(q_i)\psi(t)+\psi'(t)\,\bar{g}(q_i)-Rf(t)J_1(q_i)/q_i.$$

Following the same method as before, (3.10) is inverted to give

$$\theta = 2\sum_i\left[\left(\frac{G'(0)-\beta G(0)}{\beta-\alpha}e^{\alpha t}-\frac{G'(0)-\alpha G(0)}{\beta-\alpha}e^{\beta t}\right)+G(t)\right]\frac{J_0(rq_i)}{[J_1(q_i)]^2}, \tag{3.12}$$

the summation being extended over all the positive roots of (3.5). From the result for θ it can readily be inferred that the temperature field becomes unsteady and turbulent at the critical value of the Rayleigh number, $R_c = (2{\cdot}403)^4$. This means that the assumption of fully developed flow is not justified if the temperature gradient decreases with increases in height.

4. Particular cases

Let us suppose that

$$f(t) = EPe^{-\gamma t}, \quad \psi(t) = \hat{F}e^{-\delta t}, \quad \bar{g}(q_i) = J_1(q_i)/q_i.$$

In this case

$$\chi(t) = \frac{J_1(q_i)}{q_i} P[\hat{F} e^{-\delta t} + E(\gamma - q_i^2) e^{-\gamma t}],$$

$$F(t) = \frac{J_1(q_i)}{q_i} P \left[\frac{\hat{F} e^{-\delta t}}{(\delta+\beta)(\delta+\alpha)} + \frac{E(\gamma - q_i^2)}{(\gamma+\beta)(\gamma+\alpha)} e^{-\gamma t} \right],$$

$$F'(t) = -\frac{J_1(q_i)}{q_i} P \left[\frac{\delta \hat{F} e^{-\delta t}}{(\delta+\beta)(\delta+\alpha)} + \frac{E\gamma(\gamma - q_i^2)}{(\gamma+\beta)(\gamma+\alpha)} e^{-\gamma t} \right],$$

$$F'(0) = -P \frac{J_1(q_i)}{q_i} \left[\frac{\hat{F}\delta}{(\delta+\beta)(\delta+\alpha)} + \frac{E\gamma(\gamma - q_i^2)}{(\gamma+\beta)(\gamma+\alpha)} \right].$$

By substituting these values in (3.9) we get the expression for u_3. However, we are interested in the case when both the strength of the heat source and pressure gradient are constant. In this case

$$\chi(t) = \frac{J_1(q_i)}{q_i} P(\hat{F} - Eq_i^2), \quad F(t) = \frac{\hat{F} - Eq_i^2}{q_i^4 - R} \frac{J_1(q_i)}{q_i},$$

$$F'(t) = 0 = F'(0), \quad f'(0) = 0 = \psi'(0).$$

Hence u_3 is given by

$$u_3 = \frac{1}{\sqrt{R}} \left(\frac{\hat{F}}{2\sqrt{R}} - \frac{E}{2} \right) \left(\frac{J_0(rR^{\frac{1}{4}})}{J_0(R^{\frac{1}{4}})} - 1 \right) - \frac{1}{2} \frac{\hat{F} + E\sqrt{R}}{R} \left(1 - \frac{I_0(rR^{\frac{1}{4}})}{I_0(R^{\frac{1}{4}})} \right)$$
$$+ 2\sum_i \left[\frac{\beta(Eq_i^2 - \hat{F})}{-R + q_i^4} e^{\alpha t} + \alpha \frac{(\hat{F} - Eq_i^2)}{q_i^4 - R} e^{\beta t} \right] \frac{J_0(rq_i)}{(\beta-\alpha)} \frac{1}{J_1(q_i) q_i}. \quad (4.1)$$

As was conjectured by Morton (1960), the velocity becomes steady ultimately provided that the Rayleigh number is less than its critical value R_c, and our results agree with that of Morton if $\hat{F} = 0$. It can be readily shown that if the temperature gradient increases with height the velocity field or the temperature field can be obtained by changing R to $-R$; in this case the critical value of R is high, and the flow ultimately becomes steady. The results of Morton (1960) and Tao (1961) are particular cases of these results, and are in complete agreement.

The temperature field in this case is represented by

$$\theta = -\frac{E\sqrt{R} - \hat{F}}{2\sqrt{R}} \left(\frac{J_0(rR^{\frac{1}{4}})}{J_0(R^{\frac{1}{4}})} - 1 \right) + \frac{E\sqrt{R} + \hat{F}}{2\sqrt{R}} \left(1 - \frac{I_0(rR^{\frac{1}{4}})}{I_0(R^{\frac{1}{4}})} \right)$$
$$+ 2\sum_i \left[\beta \left(\frac{\hat{F}q_i^2 - RE}{q_i^4 - R} \right) e^{\alpha t} - \frac{\alpha(\hat{F}q_i^2 - RE)}{q_i^4 - R} e^{\beta t} \right] \frac{J_0(q_i r)}{(\alpha-\beta) q_i} \frac{1}{J_1(q_i)}. \quad (4.2)$$

If $R < R_c$, ultimately θ becomes steady. If there is no source of heat generation, i.e. $\hat{F} = 0$, θ agrees with the results of Morton. (Our θ is $1/R$ times that of Morton.)

5. Co-axial pipes

Proceeding in exactly the same way as in §4, we have

$$\bar{u}_3 = \frac{F_1'(0) - \beta F_1(0)}{\beta - \alpha} e^{\alpha t} + \frac{F_1'(0) - \alpha F_1(0)}{\alpha - \beta} e^{\beta t} + F_1(t), \quad (5.1)$$

where

$$F_1(t) = e^{\alpha t} \int^t e^{(\beta-\alpha)t} \int^t X_1(t) e^{-\beta t} (dt)^2$$

c	1·2	1·5	2·0	2·5	3·0	3·5	4·0
q_1	15·7014	6·2702	3·1230	2·0732	1·5485	1·2339	1·0244
R_c	60762	1545	95·12	18	5·75	2·31	1

TABLE 1

and $$X_1(t) = P\bar{g}_1(q_i)\psi(t)+\frac{2}{\pi q_i^2}\left[\left\{\frac{J_0(q_i c)}{J_0(q_j)}-1\right\}\{f(t)q_i^2+f'(t)\}\right],$$

$$\bar{g}_1(q_i) = \int_1^c rg(r)B_0(rq_i)\,dr,$$

where $c = b/a, a < b$,

$$B_0(rq_i) = J_0(rq_i)Y_0(q_i)-Y_0(rq_i)J_0(q_i), \tag{5.2}$$

while q_i is the ith root of the equation

$$J_0(cq_i)Y_0(q_i)-Y_0(cq_i)J_0(q_i) = 0. \tag{5.3}$$

α and β have the same meaning as before.

Equation (5.1) is inverted to give

$$u_3 = \tfrac{1}{2}\pi^2\sum_i\frac{q_i^2J_0^2(cq_i)}{J_0^2(q_i)-J_0^2(cq_i)}\left\{\frac{F_1'(0)-\beta F_1(0)}{\beta-\alpha}e^{\alpha t}+\frac{F_1'(0)-\alpha F_1(0)}{\alpha-\beta}e^{\beta t}+F_1(t)\right\}B_0(rq_i). \tag{5.4}$$

Since we are interested in the case of constant pressure gradient and constant heat source strength, we write

$$X_1(t) = \frac{2P}{\pi q_i^2}\left(\frac{J_0(q_i c)}{J_0(q_i)}-1\right)(\hat{F}-Eq_i^2),$$

$$F_1(t) = \frac{2}{\pi q_i^2}\frac{\hat{F}-Eq_i^2}{q_i^4-R}\left(\frac{J_0(q_i c)}{J_0(q_i)}-1\right) = F(0).$$

It is quite obvious that $F_1'(0) = f'(0) = 0$. Thus we get

$$u_3 = \pi\sum_i\left\{1-\frac{\alpha e^{\beta t}-\beta e^{\alpha t}}{\alpha-\beta}\right\}\frac{\hat{F}-Eq_i^2}{q_i^4-R}\frac{J_0(cq_i)}{J_0(cq_i)+J_0(q_i)}B_0(rq_i), \tag{5.5}$$

where the summation has been extended over the positive roots of (5.3). Proceeding in the same manner as in the case of velocity, we have for temperature

$$\theta = \pi\sum_i\left(1-\frac{\alpha e^{\beta t}-\beta e^{\alpha t}}{\alpha-\beta}\right)\frac{\hat{F}q_i^2-ER}{q_i^4-R}\frac{J_0(cq_i)}{J_0(cq_i)+J_0(q_i)}B_0(rq_i).$$

It is quite obvious that the solutions for velocity and temperature difference both become infinite at the smallest root of (5.3), i.e. when $R = q_i^4$, and beyond this value of R, which is the critical value R_c, the motion becomes unsteady and turbulent. The values of R_c for different values of c are given in table 1 (see Carslaw & Jaeger 1947).

It is quite clear from table 1 that the critical Rayleigh number decreases as the gap between the cylinders increases. The highest value of the Rayleigh number (when $c = 1{\cdot}2$) is $(15{\cdot}70)^4 \simeq 60762$ and R decreases to nearly 1 when $c = 4$. It can be conjectured that the Rayleigh number can be made as large as we choose by decreasing the gap between the two cylinders.

6. Unsteady flow within a hollow elliptical cylinder

In this section the same problem is discussed for the case of the elliptical tube. The fundamental equations in non-dimensional form are given by

$$\left.\begin{aligned} \frac{\partial u_3}{\partial t} &= -\phi(t) + \frac{2P}{h^2}\left(\frac{\partial^2 u_3}{\partial \xi^2} + \frac{\partial^2 u_3}{\partial \eta^2}\right)\frac{1}{\cosh 2\xi - \cos 2\eta} + P\theta, \\ \frac{\partial \theta}{\partial t} &= Ru_3 + \frac{2}{h^2(\cosh 2\xi - \cos 2\eta)}\left(\frac{\partial^2 \theta}{\partial \xi^2} + \frac{\partial^2 \theta}{\partial \eta^2}\right) + \psi(t)g(\xi,\eta), \end{aligned}\right\} \tag{6.1}$$

where $x_1 = h\cosh\xi\cos\eta$, $y_1 = h\sinh\xi\sin\eta$, $h = (1-\sigma^2)^{\frac{1}{2}} = e$, $\sigma = h\sinh\xi_0 = b/a$, $h\cosh\xi_0 = 1$ and $\xi = \xi_0$ represents the boundary of the ellipse. The boundary conditions in elliptic co-ordinates become

$$\left.\begin{aligned} u_3(\xi,\eta,t) &= 0 \quad \text{for} \quad t > 0 \quad \text{on} \quad \xi = \xi_0, \quad 0 \leqslant \eta \leqslant 2\pi, \\ u_3(\xi,\eta,0) &= 0 \quad \text{for} \quad t = 0^+ \text{ within the elliptic tube,} \\ \theta(\xi,\eta,t) &= 0 \quad \text{for} \quad t > 0 \quad \text{on} \quad \xi = \xi_0, \quad 0 \leqslant \eta \leqslant 2\pi, \\ \theta(\xi,\eta,0) &= 0 \quad \text{for} \quad t = 0^+ \text{ within the tube.} \end{aligned}\right\} \tag{6.2}$$

Making use of the notation due to Morse & Feshbach (1953) and the technique given in Gupta (1964) we have

$$\int_0^{\xi_0}\int_0^{2\pi} u_3(\xi,\eta,t)\,(\cosh 2\xi - \cos 2\eta)\,\mathrm{Se}_{2n}(\eta, q_{2n,m})\,\mathrm{Je}_{2n}(\xi, q_{2n,m})\,d\xi\,d\eta = \bar{u}_3(q_{2n,m}), \tag{6.3}$$

where $q_{2m,n}$ is the mth root of the equation

$$\mathrm{Je}_{2n}(\xi_0, q_{2n,m}) = 0, \tag{6.4}$$

and

$$\left.\begin{aligned} \mathrm{Se}_{2n}(\eta,q) &= \sum_{r=0}^{\infty} \mathrm{De}_{2r}^{2n}(q)\cos 2r\eta, \\ \mathrm{Je}_{2n}(\xi,q) &= \sum_{r=0}^{\infty} (-1)^n\,(\tfrac{1}{2}\pi)^{\frac{1}{2}}\,\mathrm{De}_{2r}^{2n}(q)\,J_{2r}(q^{\frac{1}{2}}\cosh\xi) \end{aligned}\right\} \tag{6.4a}$$

($q = s = 4q' = 4k^2q$, see Mclachlan (1951)). Hence it follows (see Gupta 1964) that

$$\int_0^{\xi_0}\int_0^{2\pi}\left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}\right)(u_3,\theta)\,\mathrm{Se}_{2n}(\eta,q_{2n,m})\,\mathrm{Je}_{2n}(\xi,q_{2n,m})\,d\xi\,d\eta = -\tfrac{1}{4}q_{2n,m}(\bar{u}_3,\bar{\theta}). \tag{6.5}$$

Hence we see that $\bar{u}_3$ and $\bar{\theta}$ satisfy the following equations:

$$\frac{d\bar{u}_3}{dt} + \frac{Pq_{2n,m}}{h^2}\bar{u}_3 = P\bar{\theta} - \bar{f}(t, q_{2n,m}), \tag{6.6}$$

$$\frac{d\bar{\theta}}{dt} + \frac{q_{2n,m}\bar{\theta}}{h^2} = R\bar{u}_3 + \psi(t)\,\bar{g}(q_{2n,m}), \tag{6.7}$$

where

$$\bar{f} = \int_0^{\xi_0}\int_0^{2\pi} f(t)\,(\cosh 2\xi - \cos 2\eta)\,\mathrm{Se}_{2n}(\eta,q_{2n,m})\,\mathrm{Je}_{2n}(\xi,q_{2n,m})\,d\xi\,d\eta,$$

with the initial conditions

$$\bar{u}_3 = 0 = \bar{\theta}, \quad d\bar{u}_3/dt = d\bar{\theta}/dt = 0 \quad \text{when} \quad t = 0. \tag{6.8}$$

By eliminating $\bar{\theta}$ from (6.6) and (6.7), we get

$$\frac{d^2\bar{u}_3}{dt^2} + (P+1)\frac{q_{2n,m}}{h^2}\frac{d\bar{u}_3}{dt} + P\left(\frac{q^2_{2n,m}}{h^4} - R\right)\bar{u}_3 = X(t), \tag{6.9}$$

where $$X(t) = \psi(t)\bar{g}(q_{2n,m}) - \bar{f}'(t, q_{2n,m}) - \frac{q^2_{2n,m}}{h^4}\bar{f}(t, q_{2n,m}).$$

Let us put $q_{2n,m}/h^2 = \gamma^2_{2n,m}$. Now the appropriate solution of (6.9) is given by

$$\bar{u}_3 = \frac{F_2'(0) - \beta F_2(0)}{\beta - \alpha} e^{\alpha t} + \frac{\alpha F_2(0) - F_2'(0)}{\beta - \alpha} e^{\beta t} + F_2(t), \tag{6.10}$$

where $$F_2(t) = e^{\alpha t}\int^t e^{(\beta-\alpha)t}\int^t e^{-\beta t} X(t)\,(dt)^2, \tag{6.11}$$

$$\left.\begin{aligned} \alpha &= \tfrac{1}{2}\{-(P+1)\gamma^2_{2n,m} + [(P-1)^2\gamma^4_{2n,m} + 4RP]^{\frac{1}{2}}\}, \\ \beta &= \tfrac{1}{2}\{-(P+1)\gamma^2_{2n,m} - [(P-1)^2\gamma^4_{2n,m} + 4RP]^{\frac{1}{2}}\}. \end{aligned}\right\} \tag{6.12}$$

Hence by the same technique as that given by the author (1964) inversion gives

$$u_3 = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\left[F_2(t) + \frac{F_2'(0) - \beta F_2(0)}{\beta - \alpha} e^{\alpha t} + \frac{\alpha F_2(0) - F_2'(0)}{\beta - \alpha} e^{\beta t}\right]$$
$$\times \frac{\mathrm{Se}_{2n}(\eta, q_{2n,m})\,\mathrm{Je}_{2n}(\xi, q_{2n,m})}{\pi\int_0^{\xi_0}\mathrm{Je}^2_{2n}(\xi, q_{2n,m})\,[\mathrm{Me}_{2n}(q_{2n,m})\cosh 2\xi - \Theta_{2n,m}]}, \tag{6.13}$$

where $$\Theta_{2n,m} = \int_0^{2\pi}\mathrm{Se}^2_{2n}(\eta, q_{2n,m})\cos 2\eta\,d\eta = \pi\left[D_0^{2n}D_2^{2n} + \sum_{r=0}^{\infty} D_{2r}^{(2n)}D_{2r+2}^{2n}\right]$$

and $$\mathrm{Me}_{2n}(q_{2n,m}) = \int_0^{2\pi}\mathrm{Se}^2_{2n}(\eta, q_{2n,m})\,d\eta. \tag{6.14}$$

Similarly, for $\bar{\theta}$ we get

$$\bar{\theta} = G_2(t) + \frac{G_2'(0) - \beta G_2(0)}{\beta - \alpha} e^{\alpha t} - \frac{G_2'(0) - \alpha G_2(0)}{\beta - \alpha} e^{\beta t}, \tag{6.15}$$

where $$G_2(t) = e^{\alpha t}\int^t e^{(\beta-\alpha)t}\int^t e^{-\beta t} Y_1(t)\,(dt)^2$$

and $$Y_1(t) = -R\bar{f}(t, q_{2n,m}) + P\psi(t)\gamma^2_{2n,m}\bar{g}(q_{2n,m}) - \psi'(t)\bar{g}(q_{2n,m}). \tag{6.16}$$

In the same way as in the case of the velocity field, after inversion, we get

$$\theta(\xi, \eta, t) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\left\{\frac{G_2'(0) - \beta G_2(0)}{\beta - \alpha} e^{\alpha t} - \frac{G_2'(0) - \alpha G_2(0)}{\beta - \alpha} e^{\beta t} + G_2(t)\right\}$$
$$\times \frac{\mathrm{Se}_{2n}(\eta, q_{2n,m})\,\mathrm{Je}_{2n}(\xi, q_{2n,m})}{\int_0^{\xi_0}\mathrm{Je}^2_{2n}(\xi, q_{2n,m})\,[\mathrm{Me}_{2n}\cosh 2\xi - \Theta_{2n,m}]\,d\xi}. \tag{6.17}$$

7. Transition to circular cylinder

We see that both u_3 and θ are independent of η when we consider the case of the circular cylinder of radius unity. We have $\gamma^2_{2n,m} = q_{0,m}$, $m = 1, 2, \ldots$, these being the roots of $J_0(q) = 0$. Also $e \to 0$ as $\xi \to \infty$, and $\sinh\xi \to \cosh\xi$, $h\cosh\xi \to r$, $h\sinh\xi\, d\xi \to dr$ and $\cosh 2\xi\, d\xi \to 2rh^2 dr$. Making use of these we find that

$$\mathrm{Se}_0(\eta, q_{2n,m}) \to 1, \quad \mathrm{Je}_{2n}(\xi, q_{2n,m}) \to J_0(rq).$$

Hence (6.10) and (6.14) degenerate to (3.9) and (3.11) respectively as the elliptical cylinder degenerates into a circular cylinder.

8. A particular case

If the strength of the heat source, as well as the pressure gradient, is an absolute constant, then,

$$X(t) = P(\bar{F} - \bar{E}\gamma_{2n,m}^2), \quad Y_1(t) = P(\bar{F}\gamma_{2n,m}^2 - R\bar{E}),$$

where

$$(\bar{E}, \bar{F}) = (E, \hat{F})\int_0^{\xi_0}\int_0^{2\pi} (\cosh 2\xi - \cos 2\eta)\,\mathrm{Se}_{2n}(\eta, q_{2n,m})\,\mathrm{Je}_{2n}(\xi, q_{2n,m})\,d\xi\,d\eta$$

and $$F(t) = F(0) = \frac{\bar{F} - \bar{E}\gamma_{2n,m}^2}{\gamma_{2n,m}^2 - R}, \quad G(t) = \frac{\bar{F}\gamma_{2n,m}^2 - R\bar{E}}{\gamma_{2n,m}^2 - R} = G(0),$$

whereas $\bar{F}'(t) = G'(t) = 0$ for all values of t. Thus we get

$$\bar{u}_3 = F(0) + (\beta e^{\alpha t} - \alpha e^{\beta t})F(0)/(\alpha - \beta), \tag{8.1}$$

which on inversion yields

$$u_3 = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \frac{\bar{F} - \bar{E}\gamma_{2n,m}^2}{(\gamma_{2n,m}^4 - R)} \frac{\mathrm{Se}_{2n}(\eta, q_{2n,m})\,\mathrm{Je}_{2n}(\xi, q_{2n,m})}{\int_0^{\xi_0} \mathrm{Je}_{2n}^2(\xi, q_{2n,m})\,[\mathrm{Me}_{2n}\cosh 2\xi - \Theta_{2n,m}]\,d\xi} \times \left(1 - \frac{\alpha e^{\beta t} - \beta e^{\alpha t}}{\alpha - \beta}\right). \tag{8.2}$$

Similarly, for the temperature distribution we have

$$\bar{\theta} = G(0)\left[1 - \left(\frac{\alpha e^{\beta t} - \beta e^{\alpha t}}{\alpha - \beta}\right)\right], \tag{8.3}$$

which on inversion yields

$$\theta = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} G(0)\left(1 - \frac{\alpha e^{\beta t} - \beta e^{\alpha t}}{\alpha - \beta}\right) \frac{\mathrm{Se}_{2n}(\eta, q_{2n,m})\,\mathrm{Je}_{2n}(\xi, q_{2n,m})}{\int_0^{\xi_0} \mathrm{Je}_{2n}^2(\xi, q_{2n,m})\,[\mathrm{Me}_{2n}\cosh 2\xi - \Theta_{2n,m}]\,d\xi}. \tag{8.4}$$

Now we sum the series for the steady state with the help of the results obtained in the appendix. From (8.2), (6.3) and (6.4) by virtue of the boundary conditions we have (see appendix)

$$\begin{aligned} u_3 = {} & \frac{1}{2\sqrt{R}}(E - \hat{F}/\sqrt{R})\left[1 - \sum_{n=0}^{\infty} \frac{2\pi D_0^{2n}(q')\,\mathrm{Se}_{2n}(\eta, q')\,\mathrm{Je}_{2n}(\xi, q')}{\mathrm{Je}_{2n}(\xi_0, q')\,\mathrm{Me}_{2n}(q')}\right] \\ & - \frac{1}{2\sqrt{R}}(E + \hat{F}/\sqrt{R})\left[1 - \sum_{n=0}^{\infty} \frac{2\pi D_0^{2n}(q')\,(-1)^n\,\mathrm{Se}_{2n}(\eta, -q')\,\mathrm{Je}_{2n}(\xi, -q')}{\mathrm{Je}_{2n}(\xi_0, -q')\,\mathrm{Me}_{2n}(-q')}\right] \\ & + \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \frac{e^{\alpha t}\beta F(0) - \alpha e^{\beta t}F(0)}{(\alpha - \beta)} \frac{\mathrm{Se}_{2n}(\eta, q_{2n,m})\,\mathrm{Je}_{2n}(\xi, q_{2n,m})}{\int_0^{\xi_0} \mathrm{Je}_{2n}^2(\xi, q_{2n,m})\,[\mathrm{Me}_{2n}\cosh 2\xi - \Theta_{2n,m}]\,d\xi}. \end{aligned} \tag{8.5}$$

R_c	39	39·6	60·8	112·59	248	702	1730
ξ_0	1·573	1·044	0·748	0·603	0·457	0·346	0·268
e	$(2{\cdot}5)^{-1}$	$(1{\cdot}55)^{-1}$	$(1{\cdot}31)^{-1}$	$(1{\cdot}18)^{-1}$	$(1{\cdot}10)^{-1}$	$(1{\cdot}06)^{-1}$	$(1{\cdot}04)^{-1}$
$q_{0,1}$	1	3	6	9	15	25	40

TABLE 2. $q_{0,1}$ is the first zero of (6.4) for different values of ξ.

$$\begin{aligned}\theta = &\frac{1}{2}\left(E-\frac{\hat{F}}{\sqrt{R}}\right)\left[1-\sum_{n=0}^{\infty}\frac{2\pi D_0^{2n}(q')\,\mathrm{Se}_{2n}(\eta,q')\,\mathrm{Je}_{2n}(\xi,q')}{\mathrm{Je}_{2n}(\xi_0,q')\,\mathrm{Me}_{2n}(q')}\right]\\ &+\frac{1}{2}\left(E+\frac{\hat{F}}{\sqrt{R}}\right)\left[1-\sum_{n=0}^{\infty}\frac{2\pi D_0^{2n}(-q')\,\mathrm{Se}_{2n}(\eta,-q')\,\mathrm{Je}_{2n}(\xi,-q')}{\mathrm{Je}_{2n}(\xi_0,-q')\,\mathrm{Me}_{2n}(-q')}\right]\\ &-\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{\beta G(0)\,e^{\alpha t}-\alpha G(0)\,e^{\beta t}}{\beta-\alpha}\,\frac{\mathrm{Se}_{2n}(\eta,q_{2n,m})\,\mathrm{Je}_{2n}(\xi,q_{2n,m})}{\int_0^{\xi_0}\mathrm{Je}_{2n}^2(\xi,q_{2n,m})\,[\mathrm{Me}_{2n}\cosh 2\xi-\Theta_{2n,m}]\,d\xi},\end{aligned} \tag{8.6}$$

where
$$\mathrm{Me}_{2n}(q') = \int_0^{2\pi}\mathrm{Se}_{2n}^2(\eta,q')\,d\eta = \pi[D_0^2+\ldots]$$
and $q' = \frac{1}{4}h^2\sqrt{R}$.

It is quite obvious that in this case when R reaches its critical value, i.e. $R_c = \gamma_{2n,m}^2$, the velocity as well as the temperature field becomes unsteady. By changing R to $-R$ we obtain the results when the temperature gradient is in opposite direction, and in this case, it can be readily seen that the steady-state terms are in complete agreement with that found by Dalip Singh (1965) provided that the notation of Mclachlan (1951) is used. The results we have derived are quite new.

9. General discussion of the solution

In all cases the solutions for the velocity and temperature consist of the two parts. One is the transient part and the other is the steady part. It is quite obvious that the transient part decreases as time increases (provided that R remains below the critical value). In the case when the upflow is heated the increase in the Rayleigh number is quite considerable, but in the case when the downflow is heated the solution for u_3 and θ both becomes unsteady and turbulent as the Rayleigh number increases.

For each class of pipes there is a different value of R_c (the critical Rayleigh number). Obviously for coaxial pipes the Rayleigh number increases as the gap between the cylinders decreases, and reaches 60 762, while it is only 33 in the case of the circular pipe. For an elliptical tube the critical value of the Rayleigh number increases with the increase of the ellipticity e as in table 2.

The striking feature of the solutions obtained is that the fully developed laminar flow which is found for small Rayleigh numbers becomes impossible as the Rayleigh number approaches the critical value.

Circular tube. When the pressure gradient and strength of the heat source is an absolute constant the flow ultimately becomes steady. Making use of Bessel's inequality, which states that for an orthonormal set $J_0(rq_i)$, whether closed or not, we have

$$\sum_i B_i^2 \leqslant \int_0^1 \{\phi(q_i r)\}^2 \, dr, \tag{9.1}$$

where the B_i^2 are the coefficients in the generalized Fourier's expansion of $\phi(rq_i)$ in terms of $J_0(rq_i)$ and (0, 1) is the interval of orthogonality. Evidently the transient part T in (4.1) satisfies

$$|T| \leqslant 2\sum_i \left(\frac{\beta_0(Eq_i^2-\hat{F})}{q_i^4-R}\, e^{\alpha_0 t} + \frac{\alpha_0(\hat{F}-Eq_i^2)}{q_i^4-R}\, e^{\beta_0 t}\right) \frac{J_0(rq_i)}{(\beta_0-\alpha_0)\, q_i J_1(q_i)},$$

where α_0, β_0 and q_0 are the smallest values at smallest root of (3.5). Hence, by means of Bessel's inequality,

$$|T| \leqslant \left[\frac{1}{\sqrt{R}}\left(\frac{\hat{F}}{2\sqrt{R}}-\frac{E}{2}\right)\left(\frac{J_0(rR^{\frac{1}{4}})}{J_0(R^{\frac{1}{4}})}-1\right) - \frac{1}{2}\frac{\hat{F}+E\sqrt{R}}{R}\left(1-\frac{I_0(rR^{\frac{1}{4}})}{I_0(R^{\frac{1}{4}})}\right)\right] \times \left[\frac{-\beta_0 e^{\alpha_0 t}+\alpha_0 e^{\beta_0 t}}{\alpha_0-\beta_0}\right]. \tag{9.2}$$

If $P = 1$, equations (9.2) and (4.1) give

$$u_3 \cong \frac{1}{2\sqrt{R}}\left[\left(\frac{\hat{F}}{\sqrt{R}}-E\right)\left(\frac{J_0(rR^{\frac{1}{4}})}{J_0(R^{\frac{1}{4}})}-1\right) - \frac{1}{2}\frac{\hat{F}+E\sqrt{R}}{R}\left(1-\frac{I_0(rR^{\frac{1}{4}})}{I_0(R^{\frac{1}{4}})}\right)\right] \times \left\{1-\frac{1}{2}\left[-\frac{(q_0^2-\sqrt{R})}{\sqrt{R}}\, e^{-(\sqrt{R}+q_0^2)t} + \frac{\sqrt{R}+q_0^2}{\sqrt{R}}\, e^{-(-\sqrt{R}+q_0^2)t}\right]\right\}. \tag{9.3}$$

Following the same method as for (9.3) we have for the temperature difference

$$\theta \cong \left[-\frac{-E\sqrt{R}-\hat{F}}{2\sqrt{R}}\left\{\frac{J_0(rR^{\frac{1}{4}})}{J_0(R^{\frac{1}{4}})}-1\right\} + \frac{E\sqrt{R}+\hat{F}}{2\sqrt{R}}\left\{\left(1-\frac{I_0(rR^{\frac{1}{4}})}{I_0(R^{\frac{1}{4}})}\right)\right\}\right] \times \left[1-\frac{1}{2}\left\{\frac{q_0^2+\sqrt{R}}{\sqrt{R}}\, e^{-(q_0^2-\sqrt{R})t} - \frac{q_0^2-\sqrt{R}}{\sqrt{R}}\, e^{-(q_0^2+\sqrt{R})t}\right\}\right]. \tag{9.4}$$

It is quite evident from (9.3) and (9.4) that as R increases the transient part dies out slowly and if it reaches its critical value or passes beyond this then both the velocity and temperature fields become unsteady and non-laminar.

Elliptical cylinder Following the same method as in the cases of the circular cylinder we have from (8.5) and (8.6)

$$u_3 \cong \left\{\frac{1}{2\sqrt{R}}(E-\hat{F}/\sqrt{R})\left[1-\sum_{n=0}^{\infty}\frac{2\pi D_0^{2n}(q')\,\mathrm{Se}_{2n}(\eta,q')\,\mathrm{Je}_{2n}(\xi,q')}{\mathrm{Je}_{2n}(\xi_0,q')\,\mathrm{Me}_{2n}(q')}\right] - \frac{1}{2\sqrt{R}}\left(E+\frac{\hat{F}}{\sqrt{R}}\right)\left[1-\sum_{n=0}^{\infty}\frac{2\pi D_0^{2n}(-q')\,\mathrm{Se}_{2n}(\eta,-q')\,\mathrm{Je}_{2n}(\xi,-q')}{\mathrm{Je}_{2n}(\xi_0,-q')\,\mathrm{Me}_{2n}(-q')}\right]\right\} \times \left\{1-\frac{1}{2}\left[-\frac{q_{0,1}/h^2-\sqrt{R}}{\sqrt{R}}\, e^{-(q_{0,1}/h^2+\sqrt{R})t} + \frac{q_{0,1}+\sqrt{R}h^2}{\sqrt{R}h^2}\, e^{-(q_{0,1}/h^2-\sqrt{R})}\right]\right\} \tag{9.5}$$

and similarly,

$$\theta \simeq \left\{\tfrac{1}{2}(E-\hat{F}/\sqrt{R})\left[1-\sum_{n=0}^{\infty}\frac{2\pi D_0^{2n}(q')\,\mathrm{Se}_{2n}(\eta,q')\,\mathrm{Je}_{2n}(\xi,q')}{\mathrm{Je}_{2n}(\xi_0,q')\,\mathrm{Me}_{2n}(q')}\right]\right.$$

$$+\frac{1}{2}\left(E+\frac{F}{\sqrt{R}}\right)\left[1-\sum_{n=0}^{\infty}\frac{2\pi D_0^{2n}(-q')\,\mathrm{Se}_{2n}(\eta-q')\,\mathrm{Je}_{2n}(\xi,-q')}{\mathrm{Je}_{2n}(\xi_0,-q')\mathrm{Me}_{2n}(-q')}\right]\Bigg\}$$

$$\times\left\{1-\frac{1}{2}\left[\frac{(q_{0,1}/h^2)+\sqrt{R}}{\sqrt{R}}e^{-(q_{0,1}/h^2-\sqrt{R})t}-\frac{(q_{0,1}/h^2)-\sqrt{R}}{\sqrt{R}}e^{-(q_{0,1}/h^2+\sqrt{R})t}\right]\right\}. \quad (9.6)$$

It can be easily verified from (9.5) and (9.6) that the transient part as in the case of the circular cylinder dies out slowly when R increases and if it reaches the critical value or beyond that the fields (temperature or the velocity) in both the cases become non-laminar, unsteady and continue to be so.

10. Convection when the pipe temperature decreases with height

In this section the case of constant pressure gradient and heat source strength is discussed in detail for both circular and elliptical tubes. Various non-dimensional quantities have been found. A general formula for the Nusselt number in the case of the unsteady flow has been derived for both types of tube in these cases. Our results to first order are in complete agreement with that of Morton for $\hat{F} = 0$ (he has not discussed the case with a heat source). Velocity distributions and temperature distributions are given in figures 1–4.

The steady cooling of ascending hot fluid or the steady heating of descending cold fluid correspond when R is positive. The general solution is as given by (4.1) and (4.2) for the case of the circular tube and by (8.5) and (8.6) in the case of the elliptical tube. We shall find various non-dimensional quantities for the two cases.

Circular tube. When the pressure gradient and the heat source strength are absolute constants we have the following.

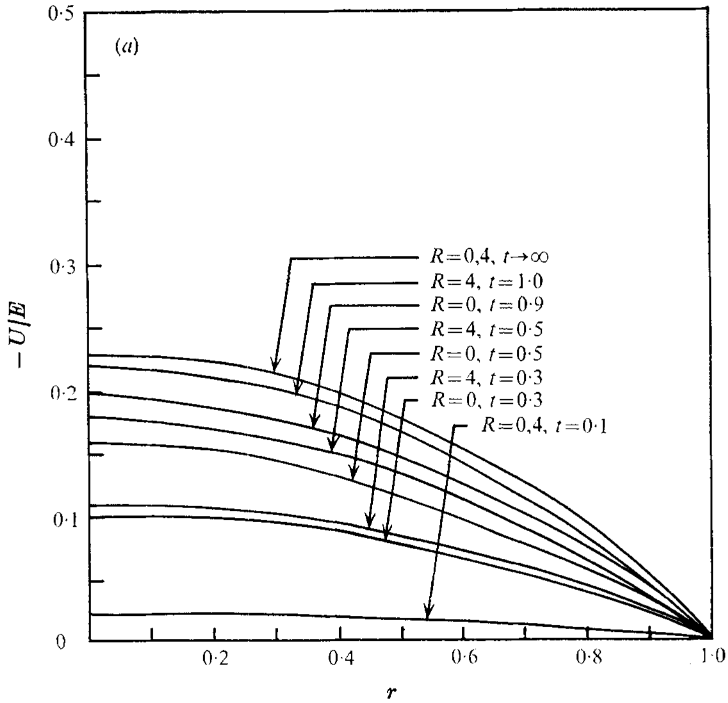
(*a*) The rate of heat transfer through the pipe walls to the fluid per unit area of pipe surface is

$$q = K\left(\frac{\partial T}{\partial r}\right)_{r=a} = K\frac{k\nu}{\alpha g a^4}\left(\frac{\partial\theta}{\partial r}\right)_{r=1} = \frac{K\beta'}{R}\left(\frac{\partial\theta}{\partial r}\right)_{r=1}$$

$$= \frac{K\beta'}{R}\left[\frac{\hat{F}-E\sqrt{R}}{2(R)^{\frac{1}{4}}}\times\frac{J_0'(R^{\frac{1}{4}})}{J_0(R^{\frac{1}{4}})}-\frac{\hat{F}+E\sqrt{R}}{2R^{\frac{1}{4}}}\times\frac{I_0'(R^{\frac{1}{4}})}{I_0(R^{\frac{1}{4}})}\right.$$

$$\left.-2\sum_i\left\{\beta\left(\frac{\hat{F}q_i^2-RE}{q_i^4-R}\right)e^{\alpha t}-\frac{\alpha(\hat{F}q_i^2-RE)}{q_i^4-R}e^{\beta t}\right\}\frac{1}{\alpha-\beta}\right], \quad (10.1)$$

where the primes imply differentiation with respect to the argument and K is the thermal conductivity. If R is less than the critical value R_c, and $\hat{F} = 0$ and $t\to\infty$, the above results agree with that of Morton. In this case θ is $1/R$ times that of Morton.



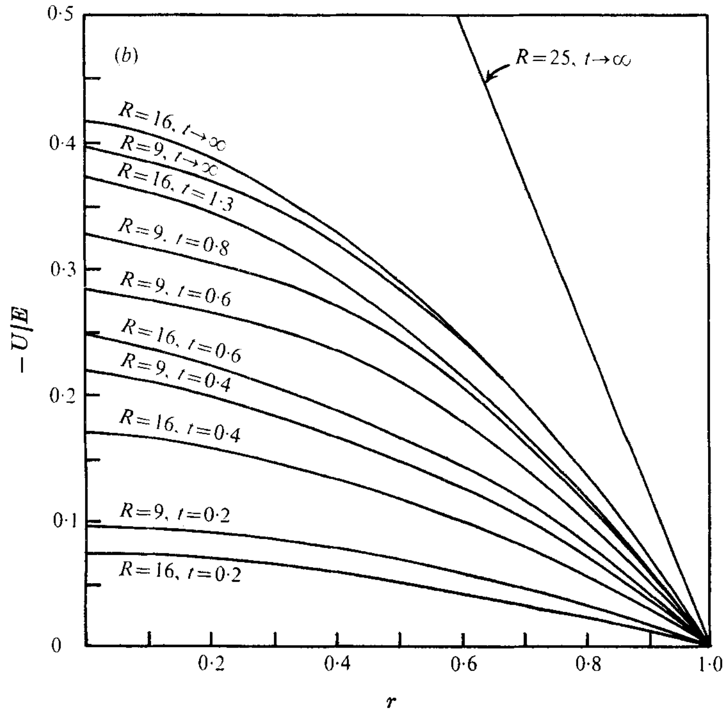


FIGURE 1. Non-dimensional velocity profiles for falling convection under a specified pressure gradient and heat source strength for a circular pipe.

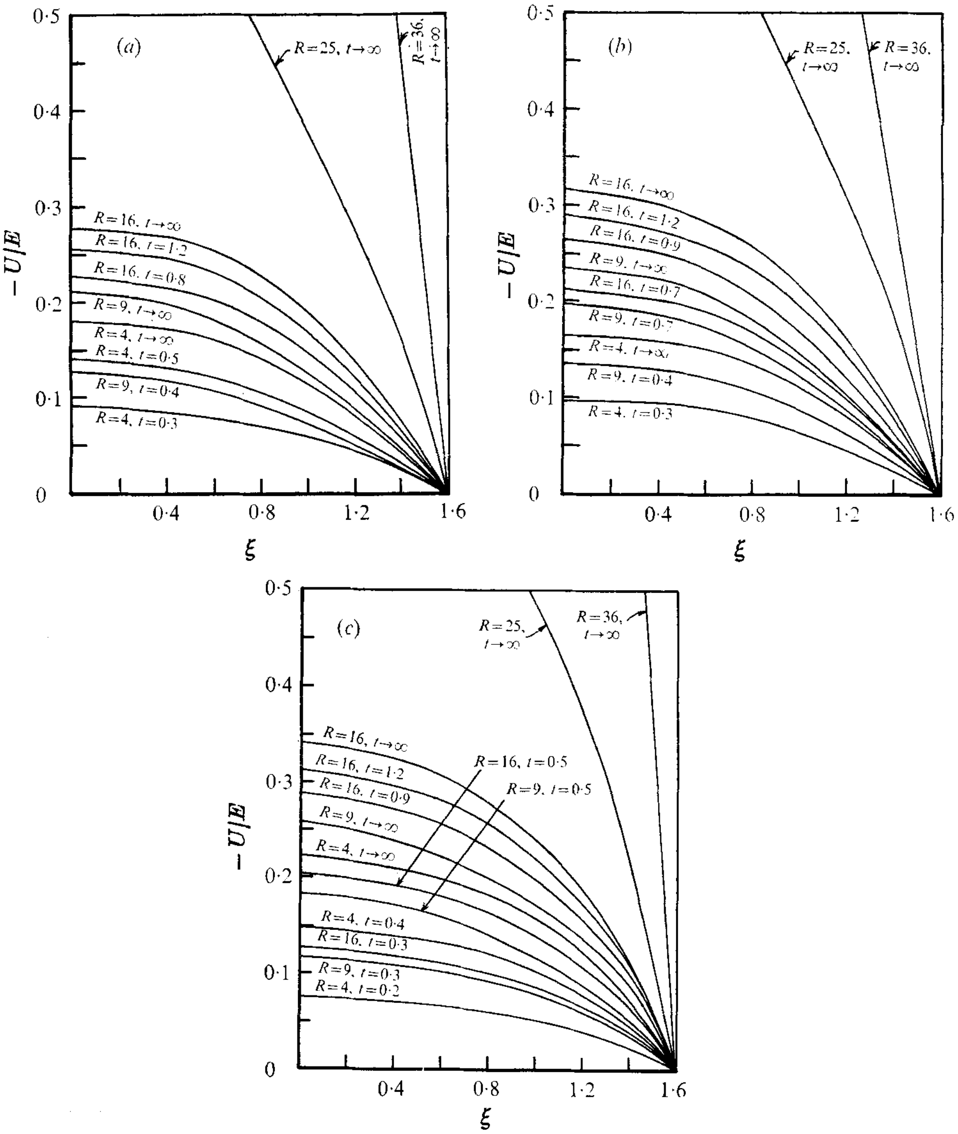


FIGURE 2. Non-dimensional velocity profiles for falling convection with a specified pressure gradient and heat source strength for an elliptical pipe.

$(a)\ \eta = 0, \quad (b)\ \eta = \tfrac{1}{4}\pi, \quad (c)\ \eta = \tfrac{1}{2}\pi.$

(*b*) The rate of volume flow through the pipe is

$$\int_0^a 2\pi r u\, dr = 2\pi a k \int_0^1 u_3 r\, dr$$

$$= 2\pi a k \left[\left\{\left(\frac{\hat{F}}{2R} - \frac{E}{2\sqrt{R}}\right)\left(\frac{J_1(R^{\frac{1}{4}})}{R^{\frac{1}{4}} J_0(R^{\frac{1}{4}})} - \frac{1}{2}\right) - \frac{\hat{F} + E\sqrt{R}}{2R}\left(\frac{1}{2} - \frac{I_1(R^{\frac{1}{4}})}{R^{\frac{1}{4}} I_0(R^{\frac{1}{4}})}\right)\right\}\right.$$

$$\left. + 2\sum_i \left\{\frac{\beta(Eq_i^2 - \hat{F})}{(q_i^4 - R)} e^{\alpha t} + \alpha\left(\frac{Eq_i^2 - \hat{F}}{R - q_i^4}\right) e^{\beta t}\right\} \frac{1}{(\alpha - \beta) q_i^2}\right]. \quad (10.2)$$

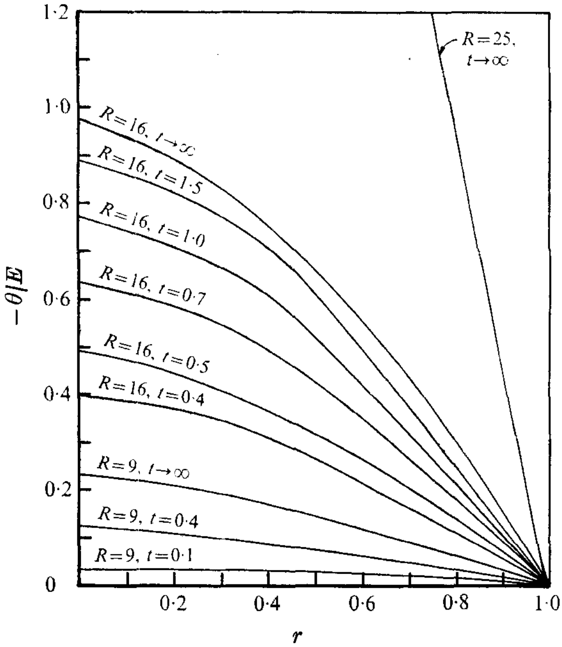


FIGURE 3. Non-dimensional buoyancy profiles for falling convection in a heated circular pipe with a constant heat source strength.

If $R < R_c$ and $t \to \infty$, the steady-state flux is

$$2\pi ak\left[\left(\frac{\hat{F}}{2R}-\frac{E}{2\sqrt{R}}\right)\left(\frac{J_1(R^{\frac{1}{4}})}{R^{\frac{1}{4}}J_0(R^{\frac{1}{4}})}-\frac{1}{2}\right)-\frac{\hat{F}+E\sqrt{R}}{2R}\left(\frac{1}{2}-\frac{I_1(R^{\frac{1}{4}})}{R^{\frac{1}{4}}I_0(R^{\frac{1}{4}})}\right)\right], \tag{10.3}$$

which is a quite new result.

If $\hat{F} \to 0$, our result is in complete agreement with that of Morton.

(*c*) A measure of the effectiveness of heat transfer is provided by the Nusselt number

$$N = \frac{\text{Rate of heat transfer rate per unit area of pipe wall} \times \text{pipe diameter}}{k \times \text{characteristic temperature in main direction of conduction}}.$$

This temperature can be measured in different ways; here, partly for comparison with the results quoted above, it will be taken as the difference between the wall temperature and the mean temperature, which is

$$T_{\text{mean}} = \frac{1}{\pi a^2}\int_0^a 2\pi r'\theta' dr' = \frac{2k\nu}{\alpha g a^3}\int_0^1 r\theta\, dr = \frac{2\beta' a}{R}\int_0^1 r\theta\, dr$$

$$= \frac{2\beta' a}{R}\left[\frac{\hat{F}-E\sqrt{R}}{2\sqrt{R}}\left(\frac{J_1(R^{\frac{1}{4}})}{R^{\frac{1}{4}}J_0(R^{\frac{1}{4}})}-\frac{1}{2}\right)+\frac{E\sqrt{R}+\hat{F}}{2\sqrt{R}}\left(\frac{1}{2}-\frac{I_1(R^{\frac{1}{4}})}{R^{\frac{1}{4}}I_0(R^{\frac{1}{4}})}\right)\right.$$

$$\left.+2\sum_i\left\{\beta\frac{\hat{F}q_i^2-RE}{q_i^4-R}e^{\alpha t}-\frac{\alpha(\hat{F}q_i^2-RE)}{q_i^4-R}e^{\beta t}\right\}\frac{1}{(\alpha-\beta)q_i^2}\right]. \tag{10.4}$$

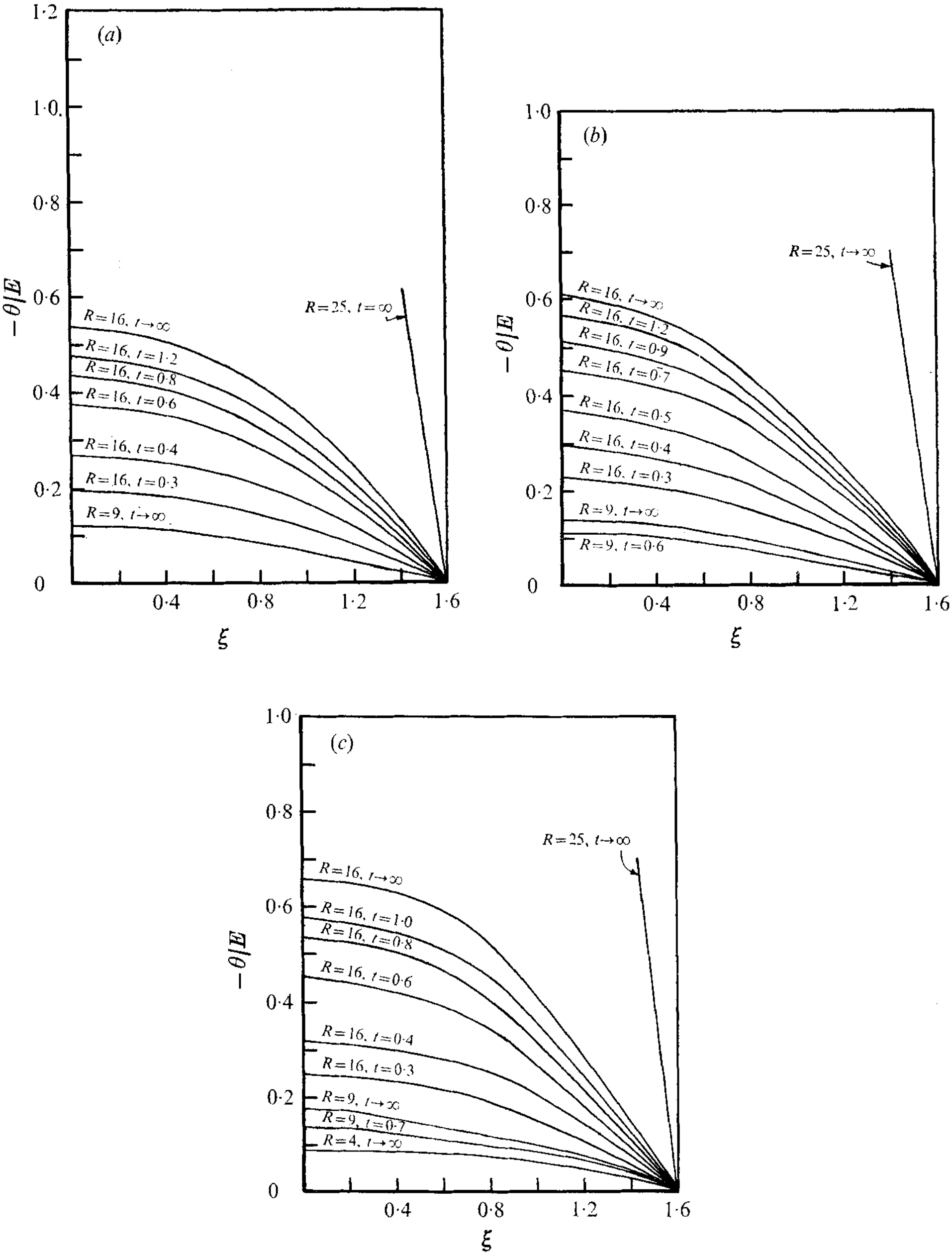


FIGURE 4. Non-dimensional buoyancy profiles for falling convection in a heated elliptical pipe with a constant heat source strength.
(a) $\eta = 0$, (b) $\eta = \frac{1}{4}\pi$, (c) $\eta = \frac{1}{2}\pi$.

(d)
$$N = -\left[\frac{(E\sqrt{R}-\hat{F})R^{\frac{1}{4}}}{2\sqrt{R}}\frac{J_0'(R^{\frac{1}{4}})}{J_0(R^{\frac{1}{4}})}+\frac{E\sqrt{R}+\hat{F}}{2\sqrt{R}}\frac{R^{\frac{1}{4}}I_0'(R^{\frac{1}{4}})}{I_0(R^{\frac{1}{4}})}\right.$$
$$\left.+2\sum_i\left\{\left(\frac{\hat{F}q_i^2-RE}{q_i^4-R}\beta e^{\alpha t}+\frac{\alpha(\hat{F}q_i^2-RE)}{R-q_i^4}e^{\beta t}\right)\right\}\frac{1}{\alpha-\beta}\right]\Big/$$
$$\left[\frac{\hat{F}-E\sqrt{R}}{2\sqrt{R}}\left(\frac{J_1(R^{\frac{1}{4}})}{R^{\frac{1}{4}}J_0(R^{\frac{1}{4}})}-\frac{1}{2}\right)+\frac{E\sqrt{R}+\hat{F}}{2\sqrt{R}}\left(\frac{1}{2}-\frac{I_1(R^{\frac{1}{4}})}{R^{\frac{1}{4}}I_0(R^{\frac{1}{4}})}\right)\right.$$
$$\left.+2\sum_i\left\{\frac{\beta(\hat{F}q_i^2-RE)}{q_i^4-R}e^{\alpha t}-\frac{\alpha(\hat{F}q_i^2-RE)}{q_i^4-R}e^{\beta t}\right\}\frac{1}{(\alpha-\beta)q_i^2}\right)\Bigg]. \quad (10.5)$$

When $t\to\infty$, i.e. in the steady case,

$$N = -\left[\frac{E\sqrt{R}-\hat{F}}{2\sqrt{R}}R^{\frac{1}{4}}\frac{J_0'(R^{\frac{1}{4}})}{J_0(R^{\frac{1}{4}})}+\frac{E\sqrt{R}+\hat{F}}{2\sqrt{R}}\frac{I_0'(R^{\frac{1}{4}})R^{\frac{1}{4}}}{I_0(R^{\frac{1}{4}})}\right]\Big/$$
$$\left[\frac{\hat{F}-E\sqrt{R}}{2\sqrt{R}}\left(\frac{J_1(R^{\frac{1}{4}})}{R^{\frac{1}{4}}J_0(R^{\frac{1}{4}})}-\frac{1}{2}\right)+\frac{E\sqrt{R}+\hat{F}}{2\sqrt{R}}\left(\frac{1}{2}-\frac{I_1(R^{\frac{1}{4}})}{R^{\frac{1}{4}}I_0(R^{\frac{1}{4}})}\right)\right], \quad (10.6)$$

which is a quite new result. It is quite obvious that if $\hat{F}\to 0$ our results agree with that of Morton.

Elliptical tubes. We shall now find all the above quantities for the case of an elliptical tube.

(A) The rate of heat transfer through the pipe walls per unit area of a pipe surface is

$$q = \frac{K\beta'}{RS}\int_0^{2\pi}\left(\frac{\partial\theta}{\partial\xi}\right)_{\xi=\xi_0}d\eta,$$

S being the length of the perimeter of the ellipse. Hence

$$q = -\frac{K\beta'}{RS}\sum_{n=0}^{\infty}\frac{\frac{1}{2}(E-\hat{F}/\sqrt{R})\,(2\pi D_0^{2n}(q'))^2\,\mathrm{Je}_{2n}'(\xi_0,q')}{\mathrm{Je}_{2n}(\xi_0,q')\,\mathrm{Me}_{2n}(q')}$$
$$+\frac{1}{2}\sum_{n=0}^{\infty}\frac{(E+\hat{F}/\sqrt{R})\,(-1)^n\,(2\pi D_0^{2n}(-q'))^2\,\mathrm{Je}_{2n}'(\xi_0,-q')}{\mathrm{Je}_{2n}(\xi_0,-q')\,\mathrm{Me}_{2n}(-q')}$$
$$+\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\left[\frac{\beta G(0)}{\beta-\alpha}e^{\alpha t}+\frac{\alpha G(0)\,e^{\beta t}}{\alpha-\beta}\right]\frac{(2\pi D_0^{2n}(q_{2n,m}))^2\,\mathrm{Je}_{2n}'(\xi_0,q_{2n,m})}{\int_0^{\xi_0}\mathrm{Je}_{2n}^2(\xi,q_{2n,m})\,[\mathrm{Me}_{2n}\cosh 2\xi-\Theta_{2n,m}]\,d\xi}, \quad (10.7)$$

where primes denote differentiation with respect to the argument.

(B) The rate of volume flow through the pipe is given by

$$\iint u_3\,dx\,dy = \frac{kah^2}{2}\int_0^{\xi_0}\int_0^{2\pi}u_3(\cosh 2\xi-\cos 2\eta)\,d\xi\,d\eta$$
$$= \frac{kah^2}{2}\left[(E-\hat{F}/\sqrt{R})\left(\frac{1}{2\sqrt{R}}\right)\left(\pi\sinh 2\xi_0\right.\right.$$
$$\left.-\sum_{n=0}^{\infty}\frac{4\pi^2D_0^{2n}(q')\int_0^{\xi_0}[D_0^{2n}(q')\cosh 2\xi-\frac{1}{2}D_2^{2n}(q')]\,\mathrm{Je}_{2n}(\xi,q')\,d\xi}{\mathrm{Je}_{2n}(\xi_0,q')\,\mathrm{Me}_{2n}(q')}\right)-\frac{E+\hat{F}/\sqrt{R}}{2\sqrt{R}}$$

$$\times\left(\pi \sinh 2\xi_0 - \sum_{n=0}^{\infty} \frac{(-1)^n\, 2\pi^2 D_0^{2n}(-q') \int_0^{\xi_0} (D_0^{2n}(-q')\, 2\cosh 2\xi - D_2^{2n}(-q'))\, \mathrm{Je}_{2n}(\xi, -q')\, d\xi}{\mathrm{Je}_{2n}(\xi_0, -q')\, \mathrm{Me}_{2n}(-q')}\right)$$

$$+\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\left[\frac{\beta F(0)\, e^{\alpha t} - \alpha F(0)\, e^{\beta t}}{\alpha-\beta} \frac{\int_0^{\xi_0} [D_0^{2n}(q_{2n,m})\, 2\cosh 2\xi - D_2^{2n}(q_{2n,m})]\, \mathrm{Je}_{2n}(\xi, q_{2n,m})\, d\xi}{\int_0^{\xi_0} \mathrm{Je}_{2n}^2(\xi, q_{2n,m})\, (\mathrm{Me}_{2n} \cosh 2\xi - \Theta_{2n,m})\, d\xi}\right]\Bigg]. \tag{10.8}$$

If $R < R_c$ and $t \to \infty$ the steady-state flux is

$$\tfrac{1}{2}kah^2\Bigg[\tfrac{1}{2}(E - \hat{F}/\sqrt{R})\, 1/\sqrt{R}\Bigg(\pi \sinh 2\xi_0$$

$$-\sum_{n=0}^{\infty} \frac{2\pi^2 D_0^{2n}(q') \int_0^{\xi_0} (D_0^{2n}(q')\, 2\cosh 2\xi - D_2^{2n}(q'))\, \mathrm{Je}_{2n}(\xi, q')\, d\xi}{\mathrm{Je}_{2n}(\xi_0, q')\, \mathrm{Me}_{2n}(q')}\Bigg)$$

$$-(E + \hat{F}/\sqrt{R})\frac{1}{2\sqrt{R}}\Bigg(\pi \sinh 2\xi_0$$

$$-\sum_{n=0}^{\infty} \frac{(-1)^n\, 2\pi^2 D_0^{2n}(-q') \int_0^{\xi_0} (D_0^{2n}(-q')\, 2\cosh 2\xi - D_2^{2n}(-q'))\, \mathrm{Je}_{2n}(\xi, -q')\, d\xi}{\mathrm{Je}_{2n}(\xi_0, -q')\, \mathrm{Me}_{2n}(-q')}\Bigg)\Bigg]. \tag{10.9}$$

(C) The mean temperature difference across a section of the pipe is

$$T_m = \frac{\beta' h^2 a^2}{2\pi b R} \int_0^{\xi_0}\int_0^{2\pi} \theta(\cosh 2\xi - \cosh 2\eta)\, d\xi\, d\eta$$

$$= \frac{\beta' h^2 a^2}{2\pi b R}\Bigg[\tfrac{1}{2}(E - \hat{F}/\sqrt{R})\left(\pi \sinh 2\xi_0 - \sum_{n=0}^{\infty} \frac{2\pi^2 D_0^{2n}(q') \int_0^{\xi_0} (D_0^{2n}(q')\, 2\cosh 2\xi - D_2^{2n}(q'))\, \mathrm{Je}_{2n}(\xi, q')\, d\xi}{\mathrm{Je}_{2n}(\xi_0, q')\, \mathrm{Me}_{2n}(q')}\right)$$

$$+\tfrac{1}{2}(E + \hat{F}/\sqrt{R})\left(\pi \sinh 2\xi_0 - \sum_{n=0}^{\infty} \frac{2\pi^2 D_0^{2n}(-q') \int_0^{\xi_0} (D_0^{2n}(-q')\, 2\cosh 2\xi - D_2^{2n}(-q'))\, \mathrm{Je}_{2n}(\xi, -q')\, d\xi}{\mathrm{Je}_{2n}(\xi_0, -q')\, \mathrm{Me}_{2n}(-q')}\right)$$

$$-\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\left(\frac{\beta G(0)\, e^{\alpha t} - \alpha G(0)\, e^{\beta t}}{\beta-\alpha}\right)$$

$$\times \frac{\int_0^{\xi_0} (D_0^{2n}(q_{2n,m})\, 2\cosh 2\xi - D_2^{2n}(q_{2n,m}))\, \mathrm{Je}_{2n}(\xi, q_{2n,m})\, d\xi}{\int_0^{\xi_0} \mathrm{Je}_{2n}^2(\xi, q_{2n,m})\, (\mathrm{Me}_{2n} \cosh 2\xi - \Theta_{2n,m})\, d\xi}\Bigg]. \tag{10.10}$$

When $t\to\infty$ the mean temperature in the steady case becomes

$$\frac{\beta' h^2 a^2}{2\pi bR}\left[\tfrac{1}{2}(E-\hat{F}/\sqrt{R})\left(\pi\sinh 2\xi_0 - \sum_{n=0}^{\infty}\frac{2\pi^2 D_0^{2n}(q')\int_0^{\xi_0}(D_0^{2n}(q')\,2\cosh 2\xi - D_2^{2n}(q'))\,\mathrm{Je}_{2n}(\xi,q')\,d\xi}{\mathrm{Je}_{2n}(\xi_0,q')\,\mathrm{Me}_{2n}(q')}\right)\right.$$
$$\left.-\tfrac{1}{2}(E+\hat{F}/\sqrt{R})\left(\pi\sinh 2\xi_0 - \sum_{n=0}^{\infty}\frac{2\pi^2 D_0^{2n}(-q')\int_0^{\xi_0}(D_0^{2n}(-q')\,2\cosh 2\xi - D_2^{2n}(-q'))\,\mathrm{Je}_{2n}(\xi,-q')}{\mathrm{Je}_{2n}(\xi_0,-q')\,\mathrm{Me}_{2n}(-q')}\right)\right]. \tag{10.11}$$

(*D*) The Nusselt number in this case is given by

$$N = -\frac{2aK\beta'}{RS}\left[\sum_{n=0}^{\infty}\frac{\tfrac{1}{2}(E-\hat{F}/\sqrt{R})\,(2\pi D_0^{2n}(q'))^2\,\mathrm{Je}'_{2n}(\xi_0,q')\,\pi}{\mathrm{Je}_{2n}(\xi_0,q')\mathrm{Me}_{2n}(q')}\right.$$
$$\left.+\tfrac{1}{2}(E+\hat{F}/\sqrt{R})\sum_{n=0}^{\infty}\frac{(-1^n)\,(2\pi D_0^{2n}(-q'))^2\,\mathrm{Je}'_{2n}(\xi_0,q')}{\mathrm{Je}_{2n}(\xi_0,-q')\,\mathrm{Me}_{2n}(-q')}\right]$$
$$+\left[\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{\beta G(0)\,e^{\alpha t}-\alpha G(0)\,e^{\beta t}}{\beta-\alpha}\times\frac{\mathrm{Je}'_{2n}(\xi_0,q_{2n\ m})\,(2\pi D_0^{2n}(q_{2n,m}))^2}{\int_0^{\xi_0}\mathrm{Je}_{2n}^2(\xi,q_{2n,m})\,(\mathrm{Me}_{2n}\cosh 2\xi-\Theta_{2n,m})\,d\xi}\right]\Bigg/$$
$$\frac{k\beta' h^2 a^2}{2\pi bR}\left[\tfrac{1}{2}(E-\hat{F}/\sqrt{R})\left(\pi\sinh 2\xi_0 - \sum_{n=0}^{\infty}\frac{2\pi^2 D_0^{2n}(q')\int_0^{\xi_0}(2D_0^{2n}(q')\cosh 2\xi - D_2^{2n}(q'))\,\mathrm{Je}_{2n}(\xi,q')\,d\xi}{\mathrm{Je}_{2n}(\xi_0,q')\,\mathrm{Me}_{2n}(q')}\right)\right.$$
$$-\tfrac{1}{2}(E+\hat{F}/\sqrt{R})\left(\pi\sinh 2\xi_0 - \sum_{n=0}^{\infty}\frac{2\pi^2 D_0^{2n}(-q')\int_0^{\xi_0}(2D_0^{2n}(-q')\cosh 2\xi - D_2^{2n}(-q'))\,\mathrm{Je}_{2n}(\xi,-q')\,d\xi}{\mathrm{Je}_{2n}(\xi_0,-q')\,\mathrm{Me}_{2n}(-q')}\right)$$
$$\left.-\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{\beta G(0)\,e^{\alpha t}-\alpha G(0)\,e^{\beta t}}{\beta-\alpha}\times\frac{\int_0^{\xi_0}\{2D_0^{2n}(q_{2n,m})\cosh 2\xi - D_2^{2n}(q_{2n,m})\}\,\mathrm{Je}_{2n}(\xi,q_{2n,m})\,d\xi}{\int_0^{\xi_0}\mathrm{Je}_{2n}^2(\xi,q_{2n,m})\,(\mathrm{Me}_{2n}\cosh 2\xi-\Theta_{2n,m})\,d\xi}\right]. \tag{10.12}$$

(*E*) *Transition to circular cylinder.* It can be readily seen that as $\xi \to \infty$ and $h \to 0$, the elliptical cylinder degenerates into a circular cylinder and all the results mentioned above agree with the corresponding results for the circular pipe.

Appendix

To sum the steady-state terms in § 8 the following procedure has been adopted. When we are dealing with the steady-state problem the equations are transformed into

$$\nabla^2 u_3 + \theta = E, \tag{A 1}$$

$$\nabla^2 \theta + R u_3 = -\hat{F}, \tag{A 2}$$

where E and $\hat{F}$ are absolute constants defined as before. From the above equations we derive the following two equations

$$(\nabla^2 + \sqrt{R})\,\phi = E\sqrt{R} - \hat{F},$$

$$(\nabla^2 - \sqrt{R})\phi_1 = -(\hat{F} + E\sqrt{R}),$$

where $\phi = \theta + \sqrt{R} u_3$ and $\phi_1 = \theta - u_3\sqrt{R}$. The boundary conditions give $\phi = 0$ and $\phi_1 = 0$ on $\xi = \xi_0$. The solution of the equations under these conditions in elliptical co-ordinates is

$$u_3 = \frac{1}{2\sqrt{R}}\left(E - \frac{\hat{F}}{\sqrt{R}}\right)\left[1 - \sum_{n=0}^{\infty} \frac{2\pi D_0^{2n}(q')\,\mathrm{Se}_{2n}(\eta, q')\,\mathrm{Je}_{2n}(\xi, q')}{\mathrm{Je}_{2n}(\xi_0, q')\,\mathrm{Me}_{2n}(q')}\right]$$
$$- \frac{1}{2\sqrt{R}}\left(E + \frac{\hat{F}}{\sqrt{R}}\right)\left[1 - \sum_{n=0}^{\infty} \frac{2\pi D_0^{2n}(-q')\,\mathrm{Se}_{2n}(\eta, -q')\,\mathrm{Je}_{2n}(\xi, -q')}{\mathrm{Je}_{2n}(\xi_0, -q')\,\mathrm{Me}_{2n}(-q')}\right]$$

and $$\theta = \tfrac{1}{2}(E - \hat{F}/\sqrt{R})\left[1 - \sum_{n=0}^{\infty} \frac{2D_0^{2n}(q')\,\mathrm{Se}_{2n}(\eta, q')\,\mathrm{Je}_{2n}(\xi, q')}{\mathrm{Je}_{2n}(\xi_0, q')\,\mathrm{Me}_{2n}(q')}\right]$$
$$+ \tfrac{1}{2}(E + \hat{F}/\sqrt{R})\left[1 - \sum_{n=0}^{\infty} \frac{(-1)^n 2D_0^{2n}(-q')\,\mathrm{Se}_{2n}(\eta, -q')\,\mathrm{Je}_{2n}(\xi, -q')}{\mathrm{Je}_{2n}(\xi_0, -q')\,\mathrm{Me}_{2n}(-q')}\right],$$

where $4q' = h^2\sqrt{R}$.

Now we use the transform defined in § 6 so that we may be able to find the transforms of u_3 and θ and their inversions. Multiplying (A 1) and (A 2) by $\mathrm{Se}_{2n}(\eta, q_{2n,m})\,\mathrm{Je}_{2n}(\xi, q_{2n,m})$ (where $q_{2n,m}$ is one of the positive roots of $\mathrm{Je}_{2n}(\xi_0, q) = 0$) and integrating with respect to η from 0 to 2π and with respect to ξ between 0 and ξ_0, we have, as in § (6.1),

$$q_{2n,m}\bar{u}_3/h^2 = \bar{\theta} - \bar{E}, \quad q_{2n,m}\bar{\theta}/h^2 = R\bar{u}_3 + \bar{F}. \tag{A 3}$$

On solving the above equations, we get

$$\bar{u}_3 = \frac{\bar{F} - \bar{E}\gamma_{2n,m}^2}{\gamma_{2n,m}^4 - R}, \quad \bar{\theta} = \frac{-\bar{E} + \bar{F}\gamma_{2n,m}^2}{\gamma_{2n,m}^4 - R}. \tag{A 4}$$

It is quite obvious that the values of $\bar{\theta}$ and $\bar{u}_3$ found above satisfy (A 1) and (A 2) hence the transforms of u_3 and θ are given by (A 4) above which are the same as those obtained in § 8. Hence inversion gives the sum of the series for the steady state.

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